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ON AN AXISYMMETRIC BOUNDARY VALUE PROBLEM FOR AN ELASTIC DIELECTRIC HALF-SPACE

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Abstract—Hankel transforms are used to construct a closed-form solution for the axisymmetric boundary value problem of a point charge forced normal to the surface of an isotropic elastic dielectric semi-space. Exact closed form expressions are obtained for the components of displacement and polarization vectors in terms of the Bessel functions and the fundamental solutions $(1/R)$, (e^{-mR}/R), R being the distance from the source point. The electric potential fields are determined both inside and outside the elastic dielectric semi-space. In the absence of electric polarization effects, the problem reduces to the classical axisymmetric Boussinesq problem of a point force applied normal to the surface of an isotropic elastic semi-space. The expressions of the components of displacements derived from this particular case are found to agree with known results

1. INTRODUCTION

Elastic dielectric materials exhibit linear piezoelectric effects and have become of importance in modern technology because of their use in the analysis and design of crystal oscillators, filters and transducers [1]. Classical phenomenological theory of peizoelectricity is concerned with the interaction between the strain tensor and the electric or polarization vector and is not derivable as the long wave limit from the modern theories of lattices of electrically polarizable atoms [2], (as classical elasticity theory can be derived from the Born–von-Karmon theory of monoatomic lattices of mass points as a long wave limit). This discrepancy in the continuum theory of classical piezoelectricity has recently been observed by Mindlin [3] and eliminated by adding to the stored energy of deformation and polarization a functional dependence on the polarization gradient. The new mathematical theory has interesting novel properties and amongst others, it predicts the existence of surface energy of deformation and polarization which has been measured in the laboratory [4] and calculated on the basis of atomic considerations [5].

Due to the nature of the equations of equilibrium very few boundary value problems have been solved within the framework of Mindlin's theory. The authors [6] used the method of images and Hankel transforms to study the problem of a point charge placed at a finite distance beneath its surface. Schwartz [7] constructed Papkovitch functions for Mindlin's theory analogous to those of classical elasticity and used these to solve the problem of a concentrated force in an infinite elastic dielectric continuum. A singular integral formulation of the boundary value problems was established by the authors [8] using the discontinuity theorems of single and double layer potentials.

In this paper, the axisymmetric boundary value problem of a point charge forced normal to the surface of an elastic dielectric semi-space is solved by the method of Hankel transforms. Exact closed form solutions are constructed for the components of the displacement and polarization vectors in terms of the Bessel functions and the fundamental solutions $1/R$ and (e^{-mR}/R), R being the distance from the source point. The potential fields are determined both inside and outside the elastic dielectric semi-space. For the case when the electric effects are absent the problem is reduced to the classical Boussinesq problem of a point force applied normal to the surface of an isotropic semispace and the components of the displacement vector derived are found to agree with known results [9].

2 BASIC EQUATIONS

For a homogeneous isotropic elastic dielectric semi-space, referred to an axisymmetric cylindrical polar coordinate system (r, θ, z), the components of the displacement vector $\mathbf{u}$, the polarization vector $\mathbf{P}$ and the potential field ϕ assume the form (u_r, o, u_z), $(P_r, 0, P_z)$ and $\phi(r, z)$, respectively.

The equations of equilibrium are given by

$$c\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right]+c_{44}\frac{\partial^2 u_r}{\partial z^2}+(c-c_{44})\frac{\partial^2 u_z}{\partial r\partial z}+d\left[\frac{\partial^2 P_r}{\partial r^2}+\frac{1}{r}\frac{\partial P_r}{\partial r}-\frac{P_r}{r^2}\right]+d_{44}\frac{\partial^2 P_r}{\partial z^2}+(d-d_{44})\frac{\partial^2 P_z}{\partial r\partial z}=-f_r \tag{2.1}$$

$$(c-c_{44})\left[\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right]+c_{44}\left[\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right]+c\frac{\partial^2 u_z}{\partial z^2}+(d-d_{44})\left[\frac{\partial^2 P_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial P_r}{\partial z}\right]+d_{44}\left[\frac{\partial^2 P_z}{\partial r^2}+\frac{1}{r}\frac{\partial P_z}{\partial r}\right]+d\frac{\partial^2 P_z}{\partial z^2}=-f_z \tag{2.2}$$

$$d\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right]+d_{44}\frac{\partial^2 u_r}{\partial z^2}+(d-d_{44})\frac{\partial^2 u_z}{\partial r\partial z}+b\left[\frac{\partial^2 P_r}{\partial r^2}+\frac{1}{r}\frac{\partial P_r}{\partial r}-\frac{P_r}{r^2}\right]+b^*\frac{\partial^2 P_r}{\partial z^2}+(b-b^*)\frac{\partial^2 P_z}{\partial r\partial z}-\frac{\partial \phi}{\partial r}-aP_r=-E_r^o \tag{2.3}$$

$$(d-d_{44})\left[\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right]+d_{44}\left[\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right]+d\frac{\partial^2 u_z}{\partial z^2}+(b-b^*)\left[\frac{\partial^2 P_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial P_r}{\partial z}\right]+b^*\left[\frac{\partial^2 P_z}{\partial r^2}+\frac{1}{r}\frac{\partial P_z}{\partial r}\right]+b\frac{\partial^2 P_z}{\partial z^2}-\frac{\partial \phi}{\partial z}-aP_z=-E_z^o \tag{2.4}$$

$$\frac{\partial P_r}{\partial r}+\frac{1}{r}P_r+\frac{\partial P_z}{\partial z}-\epsilon_o\left[\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{\partial^2 \phi}{\partial z^2}\right]=-\rho_c,\quad \text{in } R \tag{2.5}$$

$$\nabla^2\phi_o=0,\quad \text{in } R' \tag{2.6}$$

where ϕ_o is the electric potential in the exterior vacuum R' and where $f=(f_r, o, f_z)$, $\mathbf{E}=(E_r, o, E_z)$ and ρ_c are body force vector, the electric force vector and the volume charge, respectively, and b_{12}, b_{44}, c_{12}, c_{44}, d_{12}, d_{44} are dielectric constants with

$$x=x_{12}+2x_{44}(x=b,c,d),\quad b^*=b_{44}+b_{77}. \tag{2.7}$$

Components of the stress tensor and the electric tensor are given by [6]

$$T_{rr}=d_{12}\,\mathrm{div}\,\mathbf{P}+2d_{44}\frac{\partial P_r}{\partial r}+c_{12}\,\mathrm{div}\,\mathbf{u}+2c_{44}\frac{\partial u_r}{\partial r} \tag{2.8}$$

$$T_{\theta\theta}=d_{12}\,\mathrm{div}\,\mathbf{P}+2d_{44}\frac{P_r}{r}+c_{12}\,\mathrm{div}\,\mathbf{u}+2c_{44}\frac{u_r}{r} \tag{2.9}$$

$$T_{zz}=d_{12}\,\mathrm{div}\,\mathbf{P}+2d_{44}\frac{\partial P_z}{\partial z}+c_{12}\,\mathrm{div}\,\mathbf{u}+2c_{44}\frac{\partial u_z}{\partial z} \tag{2.10}$$

$$T_{zr}=T_{rz}=d_{44}\left(\frac{\partial P_r}{\partial z}+\frac{\partial P_z}{\partial r}\right)+c_{44}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right) \tag{2.11}$$

$$E_{rr}=b_{12}\,\mathrm{div}\,\mathbf{P}+2b_{44}\frac{\partial P_r}{\partial r}+d_{12}\,\mathrm{div}\,\mathbf{u}+2d_{44}\frac{\partial u_r}{\partial r}+b_o \tag{2.12}$$

$$E_{\theta\theta}=b_{12}\,\mathrm{div}\,\mathbf{P}+2b_{44}\frac{P_r}{r}+d_{12}\,\mathrm{div}\,\mathbf{u}+2d_{44}\frac{u_r}{r}+b_o \tag{2.13}$$

$$E_{zz} = b_{12}\,\text{div}\,\mathbf{P} + 2b_{44}\frac{P_r}{r} + d_{12}\,\text{div}\,\mathbf{u} + 2d_{44}\frac{\partial u_z}{\partial z} + b_o \tag{2.14}$$

$$E_{zr} = b_{44}\left(\frac{\partial P_r}{\partial z} + \frac{\partial P_z}{\partial r}\right) + b_{77}\left(\frac{\partial P_r}{\partial z} - \frac{\partial P_z}{\partial r}\right) + d_{44}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{2.15}$$

$$E_{rz} = b_{44}\left(\frac{\partial P_r}{\partial z} + \frac{\partial P_z}{\partial r}\right) + b_{77}\left(\frac{\partial P_z}{\partial r} - \frac{\partial P_r}{\partial z}\right) + d_{44}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{2.16}$$

$$\text{div}\,\mathbf{P} = \frac{\partial P_r}{\partial r} + \frac{1}{r}P_r + \frac{\partial P_z}{\partial z} \tag{2.17}$$

$$\text{div}\,\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}u_r + \frac{\partial u_z}{\partial z}. \tag{2.18}$$

The problem

Let the plane $z = o$ of the cylindrical polar coordinate system coincide with the surface of the elastic dielectric semi-space with the positive direction of the axis of symmetry $r = o$, pointing towards its interior.

For the axisymmetric boundary value problem of elastic dielectric semi-space subjected to a charged pin normal to its surface, we shall determine u_r, u_z, P_r, P_z, ϕ, ϕ_o satisfying eqns (2.1)–(2.6) and the boundary conditions

$$T_{zz}(r, o) = N\frac{\delta(r)}{r}, \qquad T_{zr}(r, o) = 0 \tag{2.19}$$

$$P_z(r, o) = 0, \qquad P_r(r, o) = 0 \tag{2.20}$$

$$\phi(r, o) = \phi_o(r, o), \lim_{z\to o}\left[P_z - \epsilon_o\left(\frac{\partial \phi}{\partial z} + \frac{\partial \phi_o}{\partial z}\right)\right] = -\epsilon_0 F\frac{\delta(r)}{r} \tag{2.21}$$

where $\delta(r)$ is the Dirac delta function and N and F are constants.

As has been pointed out in [10] the above set of boundary conditions permits latitude in specification of boundary conditions in the sense that the polarization, $\mathbf{P}$, or the potential, ϕ, together with their derivatives may be prescribed on the boundary of the region.

3. SOLUTION BY HANKEL TRANSFORMS

In this section, we use Hankel transforms to transform the homogeneous partial differential equations system (2.1)–(2.6) and the boundary conditions (2.19)–(2.21) to a system of ordinary differential equations with constant coefficients to which the solution is obtained.

Let

$$\{\mathbf{u}_r(\xi, z), \mathbf{P}_r(\xi, z)\} = \int_o^\infty rJ_1(\xi r)\{u_r, P_r\}\,\mathrm{d}r \tag{3.1}$$

$$\{\mathbf{u}_z(\xi, z), \mathbf{P}_z(\xi, z), \boldsymbol{\phi}(\xi, z), \boldsymbol{\phi}_o(\xi, z)\} = \int_o^\infty rJ_o(\xi r)\times\{u_z, P_z, \phi, \phi_o\}\,\mathrm{d}r \tag{3.2}$$

Applying $\int_o^\infty rJ_1(\xi r)$ to eqn (2.1), (2.3) and $\int_o^\infty rJ_o(\xi r)$ to eqns (2.4)–(2.6), we obtain the following transformed system of equations

$$[c_{44}D^2 - c\xi^2]\mathbf{u}_r - (c - c_{44})\xi D\mathbf{u}_z + [d_{44}D^2 - d\xi^2]\mathbf{P}_r - (d - d_{44})\xi D\mathbf{P}_z = 0 \tag{3.3}$$

$$(c - c_{44})\xi D\mathbf{u}_r + [cD^2 - c_{44}\xi^2]\mathbf{u}_z + (d - d_{44})\xi D\mathbf{P}_r + [dD^2 - d_{44}\xi^2]\mathbf{P}_z = 0 \tag{3.4}$$

$$[d_{44}D^2 - d\xi^2]\mathbf{u}_r + [d - d_{44}]\xi D\mathbf{u}_z + [b^*D^2 - b\xi^2 - a]\mathbf{P}_r - (b - b^*)\xi D\mathbf{P}_z + \xi\boldsymbol{\phi} = 0 \tag{3.5}$$

$$(d-d_{44})\xi D\mathbf{u}_r+[dD^2-d_{44}\xi^2]\mathbf{u}_z+(b-b^*)\xi D\mathbf{P}_r+[bD^2-b^*\xi^2-a]\mathbf{P}_z-D\boldsymbol{\phi}=0 \quad (3.6)$$

$$\xi\mathbf{P}_r+D\mathbf{P}_z-\epsilon_o(D^2-\xi^2)\boldsymbol{\phi}=0, \quad \text{in } R \quad (3.7)$$

$$(D^2-\xi^2)\boldsymbol{\phi}_o=0, \quad \text{in } R' \quad (3.8)$$

where $D=\dfrac{\mathrm{d}}{\mathrm{d}z}$ and we have used the results

$$\int_o^\infty rJ_v(\xi r)\left[\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}-\frac{v^2}{r^2}f\right]\mathrm{d}r=-\xi^2\int_o^\infty rJ_v(\xi,r)f\,\mathrm{d}r \quad (3.9)$$

$$\int_o^\infty rJ_o(\xi r)\left[\frac{\partial f}{\partial r}+\frac{1}{r}f\right]\mathrm{d}r=\xi\int_o^\infty rJ_1(\xi r)f\,\mathrm{d}r \quad (3.10)$$

$$\int_o^\infty rJ_1(\xi r)\frac{\partial f}{\partial r}\mathrm{d}r=-\xi\int_o^\infty rJ_o(\xi r)f\,\mathrm{d}r. \quad (3.11)$$

The solution of the transformed homogeneous system of eqns (3.3)–(3.8) is given by

$$\mathbf{u}_r(\xi,z)=A_1\,\mathrm{e}^{-\xi z}+[(c_{44}+c)+(c_{44}-c)\xi z](A_2-A_3)\,\mathrm{e}^{-\xi z}+d\xi A_4\,\mathrm{e}^{-\zeta_1 z}-d_{44}\zeta_2A_5\,\mathrm{e}^{-\zeta_2 z} \quad (3.12)$$

$$\mathbf{u}_z(\xi,z)=A_1\,\mathrm{e}^{-\xi z}+(c_{44}-c)\xi z(A_2-A_3)\,\mathrm{e}^{-\xi z}+d\zeta_1A_4\,\mathrm{e}^{-\zeta_1 z}-d_{44}\xi A_5\,\mathrm{e}^{-\zeta_2 z} \quad (3.13)$$

$$\mathbf{P}_r(\xi,z)=2\alpha a^{-1}\xi^2A_2\,\mathrm{e}^{-\xi z}-c\xi A_4\,\mathrm{e}^{-\zeta_1 z}+c_{44}\zeta_2A_5\,\mathrm{e}^{-\zeta_2 z} \quad (3.14)$$

$$\mathbf{P}_z(\xi,z)=2\alpha a^{-1}\xi^2A_2\,\mathrm{e}^{-\xi z}-c\zeta_1A_4\,\mathrm{e}^{-\zeta_1 z}+c_{44}\xi A_5\,\mathrm{e}^{-\zeta_2 z} \quad (3.15)$$

$$\boldsymbol{\phi}(\xi,z)=2\alpha\xi A_3\,\mathrm{e}^{-\xi z}+c\epsilon_o^{-1}A_4\,\mathrm{e}^{-\zeta_1 z}, \quad \text{in } R,\ z>o \quad (3.16)$$

$$\boldsymbol{\phi}_o(\xi,z)=A_6\,\mathrm{e}^{\xi z}, \quad \text{in } R',\ z<o \quad (3.17)$$

where $A_i(i=1-6)$ are arbitrary functions of ξ to be determined from the boundary conditions (2.19)–(2.21), $\zeta_i^2=\xi^2+m_i^2$ $(i=1,2)$ and m_i^2 are given by

$$m_1{}^2=\frac{c(1+\epsilon_o a)}{\epsilon_o(bc-d^2)}, \qquad m_2{}^2=\frac{ac_{44}}{b^*c_{44}-d_{44}^2}. \quad (3.18)$$

The expressions for some of the transformed components of stress and electric tensors are given by

$$\begin{aligned}\mathbf{T}_{zz}(\xi,z)=&-2c_{44}\xi A_1\,\mathrm{e}^{-\xi z}-2\xi[2\alpha a^{-1}d_{44}\xi^2+c_{44}(c_{44}\\&+\overline{c-c_{44}}\xi z)]A_2\,\mathrm{e}^{-\xi z}+2c_{44}\xi[c_{44}+(c-c_{44})\xi z]A_3\,\mathrm{e}^{-\xi z}+2\alpha\xi^2A_4\,\mathrm{e}^{-\zeta_1 z}\end{aligned} \quad (3.19)$$

$$\begin{aligned}\mathbf{T}_{zr}(\xi,z)=&-2c_{44}\xi A_1\,\mathrm{e}^{-\xi z}-2\xi[2\alpha a^{-1}d_{44}\xi^2+c_{44}(c\\&-\overline{c-c_{44}}\xi z)]A_2\,\mathrm{e}^{-\xi z}+2c_{44}\xi(c-\overline{c-c_{44}}\xi z]A_3\,\mathrm{e}^{-\xi z}+2\alpha\xi\zeta_1A_4\,\mathrm{e}^{-\zeta_1 z}\end{aligned} \quad (3.20)$$

$$\begin{aligned}\mathbf{E}_{zz}(\xi,z)=&-2d_{44}\xi A_1\,\mathrm{e}^{-\xi z}-2\xi[2b_{44}a^{-1}\alpha\xi^2+\alpha+d_{44}c_{44}\\&-d_{44}\overline{c-c_{44}}\xi z]A_2\,\mathrm{e}^{-\xi z}+2\xi[\alpha+d_{44}c_{44}-d_{44}\overline{c-c_{44}}\xi z]A_3\,\mathrm{e}^{-\xi z}\\&+[c\epsilon_o^{-1}(1+\epsilon_o a)+2\beta\xi^2]A_4\mathrm{e}^{-\zeta_1 z}+2m_2{}^2\xi\zeta_2A_5\,\mathrm{e}^{-\zeta_2 z}+b_o\frac{\delta(\xi)}{\xi}\end{aligned} \quad (3.21)$$

$$\mathbf{E}_{zr}(\xi, z) = -2\xi d_{44} A_1 e^{-\xi z} - 2\xi(2\alpha a^{-1} b_{44}\xi^2 + d_{44}(c - \overline{c - c_{44}}\xi z)] \times A_2 e^{-\xi z} + 2\xi[c - \overline{c - c_{44}}\xi z] d_{44} A_3 e^{-\xi z} + 2\beta\xi\zeta_1 A_4 e^{-\zeta_1 z} + [2m_3{}^2\xi^2 - ac_{44}] A_4 e^{-\zeta_2 z} \tag{3.22}$$

$$\left[\mathbf{P}_z - \epsilon_o \frac{\partial\phi}{\partial z}\right] = \xi[2\alpha a^{-1}\xi A_2 e^{-\xi z} + 2\alpha\epsilon_o\xi A_3 e^{-\xi z} + c_{44} A_5 e^{-\zeta_2 z}] \tag{3.23}$$

where

$$\alpha = cd_{44} - dc_{44}, \qquad \beta = cb_{44} - dd_{44} \tag{3.24}$$

$$m_3{}^2 = d_{44}^2 - b_{44}c_{44}. \tag{3.25}$$

The boundary conditions eqns (2.19)–(2.21) lead to the following system of six algebraic equations

$$-c_{44}A_1 - (2\alpha a^{-1} d_{44}\xi^2 + cc_{44})A_2 + cc_{44}A_3 + \alpha\zeta_1 A_4 = 0 \tag{3.26}$$

$$-c_{44}A_1 - (2\alpha a^{-1} d_{44}\xi^2 + c_{44}^2)A_2 + c_{44}^2 A_3 + \alpha\xi A_4 = \frac{N}{2\xi} \tag{3.27}$$

$$2\alpha a^{-1}\xi^2 A_2 - c\xi A_4 + c_{44}\zeta_2 A_5 = 0 \tag{3.28}$$

$$2\alpha a^{-1}\xi^2 A_2 - c\zeta_1 A_4 + c_{44}\xi A_5 = 0 \tag{3.29}$$

$$2\alpha\xi A_3 + c\epsilon_o^{-1} A_4 = A_6 \tag{3.30}$$

$$2\alpha\xi^2 A_3 + c\epsilon_o^{-1}\zeta_1 A_4 = \xi A_6 + F. \tag{3.31}$$

The solution of the above system of equations is obtained and is given in Appendix A.

The transformed components of the displacement and polarization vectors and the potential fields are given by

$$\mathbf{u}_r(\xi, z) = \frac{N}{2c_{44}(c - c_{44})}\frac{1}{\xi}[c_{44} + (c_{44} - c)\xi z] e^{-\xi z} + \epsilon_o F\left[\left[-\frac{d - d_{44}}{c - c_{44}} + \frac{\delta}{c}(c_{44} - c)\xi z\right] e^{-\xi z} + \frac{d}{cm_1{}^2}\xi(\xi + \zeta_1)(e^{-\zeta_1 z} - e^{-\xi z}) + \frac{d_{44}}{c_{44}m_2{}^2}(\xi + \zeta_2)(\zeta_2 e^{-\zeta_2 z} - \xi e^{-\xi z})\right] \tag{3.32}$$

$$\mathbf{u}_z(\xi, z) = \frac{N}{2c_{44}}\frac{1}{\xi}\left[-\frac{c}{c - c_{44}} - \xi z\right] e^{-\xi z} + \epsilon_o F\left[-\delta e^{-\xi z} + \frac{d}{cm_1{}^2}(\xi + \zeta_1)(\zeta_1 e^{-\zeta_1 z} - \xi e^{-\xi z}) + \frac{d_{44}}{c_{44}m_2{}^2}(\xi + \zeta_2) \times (\xi e^{-\zeta_2 z} - \zeta_2 e^{-\xi z})\right] \tag{3.33}$$

$$\mathbf{P}_r(\xi, z) = \epsilon_o F\left[-\frac{1}{m_1{}^2}\xi(\xi + \zeta_1)(e^{-\zeta_1 z} - e^{-\xi z}) + \frac{1}{m_2{}^2}\zeta_2(\xi + \zeta_2)(e^{-\zeta_2 z} - e^{-\xi z})\right] \tag{3.34}$$

$$\mathbf{P}_z(\xi, z) = -\epsilon_o F\left[\frac{1}{m_1^2}(\xi+\zeta_1)(\zeta_1 e^{-\zeta_1 z} - \xi e^{-\xi z}) + \frac{1}{m_2^2}(\xi+\zeta_2)(\xi e^{-\zeta_2 z} - \zeta_2 e^{-\xi z})\right] \tag{3.35}$$

$$\phi = -\frac{\alpha N}{c_{44}(c-c_{44})} e^{-\xi z} + F\left[\frac{1}{m_1^2}(\xi+\zeta_1) e^{-\zeta_1 z} + \epsilon_o\left\{-\frac{2\alpha\delta}{c}\xi + \frac{a}{m_1^2}(\xi+\zeta_1) + \frac{a}{m_2^2}\zeta_2\frac{(\xi+\zeta_2)}{\xi}\right\} e^{-\xi z}\right] \tag{3.36}$$

$$\phi_o(\xi, z) = -\frac{\alpha N}{c_{44}(c-c_{44})} e^{-\xi z} + \epsilon_o F\left[-\frac{2\delta\alpha}{c}\xi + \frac{\epsilon_o^{-1}+a}{m_1^2}(\xi+\zeta_1) + \frac{a}{m_2^2}\frac{\zeta_2}{\xi}(\xi+\zeta_2)\right] e^{-\xi z}. \tag{3.37}$$

4 THE SOLUTION AND THE PARTICULAR CASE

The inverse Hankel transforms of order 1 and 0 corresponding to eqns (3.1), (3.2) are defined as

$$\{u_r(r, z), P_r(r, z)\} = \int_o^\infty \xi J_1(\xi r)\{\mathbf{u}_r, \mathbf{P}_r\}\, d\xi \tag{4.1}$$

$$\{u_z(r, z), P_z(r, z), \phi(r, z), \phi_o(r, z)\} = \int_o^\infty \xi J_o(\xi r) \{\mathbf{u}_z, \mathbf{P}_z, \boldsymbol{\phi}, \boldsymbol{\phi}_o\}\, d\xi. \tag{4.2}$$

Multiplying the eqns (3.32), (3.34) by $\int_o^\infty \xi J_1(\xi r)$; eqns (3.33), (3.35) and (3.37) by $\int_o^\infty \xi J_o(\xi r)$, integrating with respect to ξ and using the formulae given in Appendix B, we obtain

$$u_r(r, z) = \epsilon_o F\left\{\frac{1}{r}\left[\frac{d-d_{44}}{c-c_{44}}\frac{\partial}{\partial z} + \frac{\delta}{c}(c_{44}-c)z\frac{\partial^2}{\partial z^2} + \left(\frac{d}{cm_1^2}+\frac{d_{44}}{c_{44}m_2^2}\right)\frac{\partial^3}{\partial z^3}\right]\left(1-\frac{z}{R}\right)\right.$$
$$-\frac{1}{r}\left[\frac{d}{cm_1^2}\frac{\partial}{\partial z}A(-m_1)\left(e^{-m_1 z} - \frac{z}{R}e^{-m_1 R}\right) + \frac{d_{44}}{c_{44}m_2^2}\frac{\partial}{\partial z}A(-m_2)\left(e^{-m_2 z}-\frac{z}{R}e^{-m_2 R}\right)\right]$$
$$+\frac{\partial^2}{\partial z\partial r}\left[-\frac{d}{c}I_o\left(\frac{m_1}{2}\overline{R-z}\right)K_o\left(\frac{m_1}{2}\overline{R+z}\right)\right.$$
$$+\left(\frac{d}{cm_1^2}+\frac{d_{44}}{c_{44}m_2^2}\right)\frac{\partial^2}{\partial_z 2}\left[I_o\left(\frac{m^2}{2}\overline{R-z}\right)K_o\right.$$
$$\left.\left.\left(\frac{m_2}{2}R\bar{+}z\right)\right] + \frac{\partial^2}{\partial z\partial r}\left[\frac{d}{cm_1^2}A(+m_1)J(m_1) - \frac{d_{44}}{c_{44}m_2^2}A(m_2)J(m_2)\right]\right\} \tag{4.3}$$

$$u_z(r, z) = \epsilon_o F\left\{\frac{\partial}{\partial z}\left[\delta + \frac{d}{cm_1^2}\frac{\partial^2}{\partial_z 2} + \frac{d_{44}}{c_{44}m_2^2}A(+m_2)\right]\left(\frac{1}{R}\right) - \frac{\partial}{\partial z}\left[\frac{d}{cm_1^2}\frac{\partial^2}{\partial z^2}\right.\right.$$
$$\left.\times\left(\frac{e^{-m_1 r}}{R}\right) + \frac{d_{44}}{c_{44}m_2^2}A(-m_2)\left(\frac{e^{-m_2 R}}{R}\right)\right]$$
$$+\frac{\partial^2}{\partial_z 22}\left[\frac{d}{cm_1^2}A(-m_1)\left(I_o\left(\frac{m_1}{2}\overline{R-z}\right)K_o\left(\frac{m_1}{2}\overline{R+z}\right)\right.\right.$$
$$\left.\left.+\frac{d_{44}}{c_{44}m_2^2}A(-m_2)I_o\left(\frac{m_2}{2}\overline{R-z}\right)\right]K_o\left(\frac{m_2}{2}\overline{R+z}\right) - \frac{\partial^2}{\partial z^2}\left[\frac{d}{cm_1^2}A(+m_1)J(m_1)\right.$$
$$\left.\left.+\frac{d_{44}}{c_{44}m_2^2}A(+m_z)J(m_z)\right]\right\} \tag{4.4}$$

$$P_r(r,z) = -\epsilon_o F\left\{\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{m_1^2}\frac{\partial^2}{\partial z^2}-\frac{1}{m_2^2}A(+m_2)\right]\left(1-\frac{z}{R}\right)-\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{m_1^2}A(-m_1)\right.\right.$$
$$\times\left((e^{-m_1 z}-\frac{z}{R}e^{-m_1 R}\right)-\frac{1}{m_2^2}\frac{\partial}{\partial z}\left(e^{-m_2 z}-\frac{z}{R}e^{-m_2 R}\right)\Bigg]+\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{m_1^2}A(-m_1)\right.$$
$$\times\left(I_o\left(\frac{m_1}{2}\overline{R-z}\right)K_o\left(\frac{m_1}{2}\overline{R+z}\right)\right)-\frac{1}{m_2^2}\frac{\partial}{\partial z}\left(I_o\left(\frac{m_2}{2}\overline{R-z}\right)K_o\left(\frac{m_2}{2}\overline{R+z}\right)\right)\Bigg]$$
$$\left.-\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{m_1^2}A(+m_1)J(m_1)-\frac{1}{m_2^2}A(+m_2)J(m_2)\right]\right\} \tag{4.5}$$

$$P_z(r,z) = -\epsilon_o F\left\{\frac{\partial}{\partial z}\left[\frac{1}{m_1^2}\frac{\partial^2}{\partial z^2}+\frac{1}{m_2^2}A(+m_2)\right]\left(\frac{1}{R}\right)-\frac{\partial}{\partial z}\left[\frac{1}{m_1^2}\frac{\partial^2}{\partial_z 2}\left(\frac{e^{-m_1 R}}{R}\right)\right.\right.$$
$$+\frac{1}{m_2^2}A(-m_2)\left(\frac{e^{-m_2 R}}{R}\right)\Bigg]-\frac{2}{z^2}A(-m_1)\left(I_o\left(\frac{m_1}{2}\overline{R-z}\right)\right)K_o\left(\frac{m_1}{2}\overline{R+z}\right)$$
$$-\frac{1}{m_2^2}A(-m_2)\left(I_o\left(\frac{m_2}{2}\overline{R-z}\right)K_o\left(\frac{m_2}{2}\overline{R+z}\right)\right]-\frac{\partial^2}{\partial z^2}\left[\frac{1}{m_1^2}A(+m_1)J(m_1)\right.$$
$$\left.\left.-\frac{1}{m_2^2}A(+m_2)J(m_2)\right]\right\} \tag{4.6}$$

$$\phi(r,z) = \frac{F_o}{m_1^2}\frac{\partial}{\partial z}\left[-\frac{\partial}{\partial z}\left(\frac{1-e^{-m_1 R}}{R}\right)+A(-m_1)\left(I_o\left(\frac{m_1}{2}\overline{R-z}\right)K_o\left(\frac{m_1}{2}\overline{R+z}\right)\right)\right.$$
$$\left.+A(+m_1)J(m_1)\right]+\phi_o(r,z) \tag{4.7}$$

$$\phi_o(r,z) = \epsilon_o F\left[\left(\frac{\epsilon_o^{-1}+a}{m_1^2}+\frac{a}{m_2^2}-\frac{2\delta\alpha}{c}\right)\frac{\partial^2}{\partial z^2}+m_2^2\right]\left(\frac{1}{R}\right)-\frac{\partial}{\partial z}\left[\frac{\epsilon_o^{-1}+a}{m_1^2}A(+m_1)J(m_1)\right.$$
$$\left.\left.+\frac{a}{m_2^2}A(+m_2)J(m_2)\right]\right\} \tag{4.8}$$

where $A(\pm m_i)$ are the differential operators defined by

$$A(\pm m_i) = \frac{\partial^2}{\partial z^2} \pm m_i^2 \quad (i = 1, 2),$$

$J(m_i)$, $(i = 1, 2)$ are integrals given by (B9) in Appendex B and $R = \surd(r^2 + z^2)$, is the distance from the source point.

The mechanical and the electrical stresses are determined from eqns (2.8)–(2.18).

The particular case

The solution to the classical Boussinesq problem of a concentrated point force applied normal to the surface of an isotropic elastic semi-space can be derived by neglecting the electric effects. Setting

$$F = 0, \quad d_{12} = d_{44} = d = 0, \quad \alpha = 0 \tag{4.12}$$

one finds that

$$P_r(r,z) = P_z(r,z) = \phi(r,z) = \phi_o(r,z) = 0 \tag{4.13}$$

and the residual expressions for the displacement vector components are obtained as

$$u_r(r,z) = \frac{N}{2c_{44}(c-c_{44})}\frac{1}{r}\left[c_{44}-(c_{44}-c)z\frac{\partial}{\partial z}\right]\left(1-\frac{z}{R}\right) \tag{4.14}$$

$$u_z(r, z) = -\frac{N}{2c_{44}}\left[\frac{c}{c - c_{44}} - z\frac{\partial}{\partial z}\right]\left(\frac{1}{R}\right) \tag{4 15}$$

which with minor change in notation ($c_{12} = \lambda$, $c_{44} = \mu$, $c = \lambda + 2\mu$, $N = -(P/2\pi)$) agree with the known results[9]

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APPENDIX A

The solution of the algebraic system of eqns (3 26)–(3 31) is found and is given by

$$A_1 = -\frac{cN}{2c_{44}(c - c_{44})}\frac{1}{\xi} - \frac{\epsilon_o F}{c_{44}}\left[\delta c_{44} + \frac{1}{m_1^2}\left(d_{44}\xi - \frac{\alpha}{c}\zeta_1\right)(\xi + \zeta_1) + \frac{d_{44}}{m_2^2}\zeta_2(\xi + \zeta_2)\right] \tag{A1}$$

$$A_2 = \frac{\epsilon_o a F}{2\alpha}\frac{1}{\xi^2}\left[\frac{\xi}{m_1^2}(\xi + \zeta_1) + \frac{\zeta_2}{m_2^2}(\xi + \zeta_2)\right] \tag{A2}$$

$$A_3 = -\frac{N}{2c_{44}(c - c_{44})}\frac{1}{\xi} - \frac{\epsilon_o \delta F}{c} + \frac{\epsilon_o a F}{2\alpha}\frac{1}{\xi^2}\left[\frac{\xi}{m_1^2}(\xi + \zeta_1) + \frac{\zeta_2}{m_2^2}(\xi + \zeta_2)\right] \tag{A3}$$

$$A_4 = \frac{\epsilon_o F}{c m_1^2}(\xi + \zeta_1) \tag{A4}$$

$$A_5 = -\frac{\epsilon_o F}{c_{44} m_2^2}(\xi + \zeta_2) \tag{A5}$$

$$A_6 = -\frac{\alpha N}{c_{44}(c - c_{44})} + F\left[-\frac{2\epsilon_o \delta \alpha}{c}\xi + \frac{1 + \epsilon_o a}{m_1^2}(\xi + \zeta_1) + \frac{\epsilon_o a}{m_2^2}\frac{\zeta_2}{\xi}(\xi + \zeta_2)\right] \tag{A6}$$

where

$$\delta = \frac{c d_{44} - d c_{44}}{c_{44}(c - c_{44})} = \frac{\alpha}{c_{44}(c - c_{44})} \tag{A7}$$

APPENDIX B

Some useful integrals for inverse Hankel transforms

(a) Integrals involving $J_o(\xi r)$

$$\int_o^\infty \xi^n e^{-\xi z} J_o(\xi r)\, d\xi = (-1)^n \frac{\partial^n}{\partial z^n}\left(\frac{1}{R}\right) \tag{B1}$$

$$\int_o^\infty \xi^n \zeta^{-1} e^{-\xi z} J_o(\xi r)\, d\xi = (-1)^n \frac{\partial^n}{\partial z^n} J(m) \tag{B2}$$

$$\int_o^\infty \xi \zeta^{n-1} e^{-\zeta z} J_o(\xi r)\, d\xi = (-1)^n \frac{\partial^n}{\partial z^n}\left(\frac{e^{-mR}}{R}\right) \tag{B3}$$

$$\int_o^\infty \zeta^{n-1} e^{-\zeta z} J_o(\xi r)\, d\xi = (-1)^n \frac{\partial^n}{\partial z^n}\left[I_o\left(\frac{m}{2}\overline{R - z}\right) K_o\left(\frac{m}{2}\overline{R + z}\right)\right] \tag{B4}$$

(b) Integrals involving $J_1(\xi r)$

$$\int_o^\infty \xi^n \, e^{-\xi z} J_1(\xi r) \, d\xi = (-1)^n \frac{1}{r} \frac{\partial^n}{\partial z^n} \left(1 - \frac{z}{R}\right) \tag{B5}$$

$$\int_o^\infty \xi^n \zeta^{-1} \, e^{-\xi z} J_1(\xi r) \, d\xi = (-1)^n \frac{\partial^n}{\partial z^{n-1} \partial r} J(m) \tag{B6}$$

$$\int_o^\infty \xi \zeta^{n-1} \, e^{-\zeta z} J_1(\xi r) \, d\xi = (-1)^{n+1} \frac{\partial^{n+1}}{\partial z^n \partial r} \left[I_o\left(\frac{m}{2} \overline{R-z}\right) K_o\left(\frac{m}{2} \overline{R+z}\right) \right] \tag{B7}$$

$$\int_o^\infty \zeta^n \, e^{-\zeta z} J_1(\xi r) \, d\xi = (-1)^n \frac{1}{r} \frac{\partial^n}{\partial z^n} \left(e^{-mz} - \frac{z}{R} e^{-mR} \right) \tag{B8}$$

where

$$\int_o^\infty \zeta^{-1} \, e^{-\xi z} J_o(\xi r) \, d\xi = J(m) \tag{B9}$$

$$R = \sqrt{(r^2 + z^2)}, \quad \zeta = \sqrt{(\xi^2 + m^2)} \tag{B10}$$